Lecture 40

TM Example, Encoding of a TM

Turing Machines: An Example

Example: Design a TM that decides $L = \{0^n1^n2^n \mid n \geq 1\}$ over $\{0,1,2\}$.

Solution: Idea: First ensure that input initially contains a few 0s, followed by a few 1s, followed by a few 2s. Then, check that the number of 0s, 1s, and 2s are equal.

$Q = \{q_s, q_a, q_r, \ldots\}$

$\Sigma = \{0,1,2\}$ $\Gamma = \{0,1,2, \sqcup, \ldots\}$

δ : Ensuring that input contains 0s followed by 1s followed by 2s.

$\delta(q_s,0) = (q_{find_1},0,R)$, $\delta(q_s,1) = (q_r,1,R)$, $\delta(q_s,2) = (q_r,2,R)$, $\delta(q_s,\sqcup) = (q_r,\sqcup,R)$

$\delta(q_{find_1},0) = (q_{find_1},0,R)$, $\delta(q_{find_1},1) = (q_{find_2},1,R)$, $\delta(q_{find_1},2) = (q_r,2,R)$, $\delta(q_{find_1},\sqcup) = (q_r,\sqcup,R)$

$\delta(q_{find_2},0) = (q_r,0,R)$, $\delta(q_{find_2},1) = (q_{find_2},1,R)$, $(q_{find_2},2) = (q_{find_\sqcup},2,R)$, $(q_{find_2},\sqcup) = (q_r,\sqcup,R)$

$\delta(q_{find_\sqcup},0) = (q_r,0,R)$, $\delta(q_{find_\sqcup},1) = (q_r,1,R)$, $(q_{find_\sqcup},2) = (q_{find_\sqcup},2,R)$, $(q_{find_\sqcup},\sqcup) = (q_{cr_2},\sqcup,L)$ …

Turing Machines: An Example

Example: Design a TM that decides $L = \{0^n 1^n 2^n \mid n \geq 1\}$

Solution: δ : Overview of checking that input contains equal number of 0s, 1s, and 2s.

- Replace the rightmost 2 by Z, rightmost 1 by Y, and rightmost 0 by X, in one iteration, repeatedly.
- Halt with q_r if machine can replace one out of $\{0,1,2\}$ but not the others.
- Halt with q_a if machine has replaced all of $\{0,1,2\}$.

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Encoding a TM

Goal: Encode a TM M as a binary string x such that M can be recovered from x.

Warm Up: Encoding an integer:

$$3 \rightarrow 11, \quad 5 \rightarrow 101, \quad 45 \rightarrow 101101$$ *(Encode using binary representation)*

Encoding a pair (or tuple) of integers:

$$\langle 5,2 \rangle = 10110$$ *(Concatenate the binary representation of 5 and 2)*

$\langle 5,2 \rangle$ cannot be recovered from 10110 as 10110 is also the encoding of $\langle 2,6 \rangle$.

Scheme: Encode $\langle x, y \rangle$ as $rep(x)01rep(y)$, where $rep(x)$, $rep(y)$ are the binary strings obtained from replacing every 0 with 00 and every 1 with 11 in the binary representation of x, y, respectively.

$$\langle 5,2 \rangle = 110011011100$$ *(01 acts like a separator of two numbers)*

Encoding a TM

Observation: Encoding δ of Turing machine M is sufficient to encode M.

Let $f : Q \to \mathbb{Z}^+$, $g : \Gamma \to \mathbb{Z}^+$, $h : \{L, R\} \to \{1,2\}$ be injective functions from Q, Γ, and $\{L, R\}$ to integers such that:

- f maps $q_{start}, q_{accept}, q_{reject}$ to $1,2,3$, respectively.
- g maps $0,1,\sqcup$ to $1,2,3$, respectively.

We can encode one entry of δ the following way:

$$\delta(q, a) = (q', b, M) \to rep(f(q))01rep(g(q))01rep(f(q'))01rep(g(b))01rep(h(M))$$

E.g. $\delta(q_{start}, 0) = (q_{accept}, 1, R) \to 11011101110001110001100$

Complete δ can be encoded by concatenating encoding of all entries using 01 as separator.

Note: Binary strings not in required format such as 1010 correspond to some trivial TM.